Kinetic Equations

Solution to the Exercises

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Exercise 1

Let f be a continuous function on $\mathbb{R}$ (or $\mathbb{R}_+$). Assume in addition that $\lim_{|x|\to+\infty} f(x) = 0$. Prove that f is uniformly continuous on $\mathbb{R}$ (or $\mathbb{R}_+$).

Hint: One may rely on the Heine-Cantor theorem, stating that any continuous function f defined between two metric spaces E and F is uniformly continuous if E is a compact set.

Proof. We will prove the result in $\mathbb{R}_+$; one can similarly get the result in $\mathbb{R}$. Let $\varepsilon > 0$ be fixed. Given that f vanishes at infinity, we consider R such that for any $x > R$ we get $|f(x)| < \frac{\varepsilon}{2}$; now, f is continuous in $[0, R+1]$; using the Heine-Cantor theorem we deduce that f is absolutely continuous. There exists then $\delta > 0$ such that for any $x, y \in [0, R+1]$ such that $|x - y| < \delta$ we get $|f(x) - f(y)| < \varepsilon$. We can also assume without loss of generality that $\delta < 1$.

We now have that if $x, y \in [R, +\infty)$ we have $|f(x) - f(y)| \leqslant |f(x)| + |f(y)| < \varepsilon$.

Given then two points $x, y \in \mathbb{R}_+$ such that $|x - y| < \delta$ we get that either $x, y \in [0, R+1]$ or $x, y \in [R, +\infty)$. In any case we can conclude that $|f(x) - f(y)| < \varepsilon$ and therefore f is absolutely continuous.

□

Exercise 2

Let $r < r'$ be two positive real numbers. Define the function:

$$\Delta_t\left(r, r'\right) = f(t, r) - f\left(t, r'\right), \tag{1}$$

where f is a solution of the Boltzmann equation:

$$\partial_t f + L(f) f = J(f). \tag{2}$$

Assume that f satisfies

$$0 \leqslant f(t, r) \leqslant \frac{c}{(1+r)^k} \tag{3}$$

for all $(t,r) \in \mathbb{R}_+^2$, with $k > 6$, and where

$$J(f)(t,r) = 4\int_0^{+\infty}\int_0^{+\infty} f(t,u)\, f(t,v)\, G(r,u,v)\, uv du dv, \tag{4}$$

$$G(r,u,v) = \begin{cases} 0 & \text{if } u^2+v^2 \leqslant r^2, \\ 1 & \text{if } u \geqslant r,\ v \geqslant r, \\ \frac{v}{r} & \text{if } u \geqslant r,\ v \leqslant r, \\ \frac{u}{r} & \text{if } u \leqslant r,\ v \geqslant r, \\ \frac{\sqrt{u^2+v^2-r^2}}{r} & \text{if } u^2+v^2 \geqslant r^2,\ u \leqslant r,\ v \leqslant r, \end{cases} \tag{5}$$

$$L(f)(t,r) = \int_0^r \left(2r + \frac{2u^2}{3r}\right) f(t,u)\, u^2 du + \int_r^{+\infty} \left(2u + \frac{2r^2}{3u}\right) f(t,u)\, u^2 du. \tag{6}$$

(i) Prove that Δ_t solves the differential equation:

$$\partial_t \Delta_t(r,r') + L(f)(t,r)\, \Delta_t(r,r') = \tag{7}$$

$$= J(f)(t,r) - J(f)(t,r') + f(t,r')\left(L(f)(t,r') - L(f)(t,r)\right). \tag{8}$$

(ii) Prove that there exists a positive number $\rho = \rho(t,r,r') \in (r,r')$ such that (7) can be rewritten as

$$\partial_t \Delta_t(r,r') + L(f)(t,r)\, \Delta_t(r,r') = \tag{9}$$

$$= (r'-r)\left(f(t,r')\, \partial_r L(f)(t,\rho) - \partial_r J(f)(t,\rho)\right). \tag{10}$$

Hint: Introduce a suitable function in order to apply the Mean Value Theorem.

(iii) Proof (and justify carefully) that

$$\partial_r L(f)(t,r) = \int_0^r \left(2 - \frac{2u^2}{3r^2}\right) u^2 f(t,u)\, du + \frac{4r}{3}\int_0^{+\infty} u f(t,u)\, du \tag{11}$$

and

$$\partial_r J(f)(t,r) = -\frac{8}{r^2}\left(\int_0^r u^2 f(t,u)\, du\right)\left(\int_r^{+\infty} v f(t,v)\, dv\right) \tag{12}$$

$$-\frac{4}{r^2}\int_0^r u f(t,u)\int_{\sqrt{r^2-u^2}}^r \frac{u^2+v^2}{\sqrt{u^2+v^2-r^2}} v f(t,v)\, dv du. \tag{13}$$

(iv) Prove that the absolute values of the partial derivatives $|\partial_r L(f)|$ and $|\partial_r J(f)|$ can be bounded respectively by two constants $M_1, M_2 > 0$ that do not depend on t and r.

Hint: To control the second term of $\partial_r J(f)$, one can prove and use that

$$f(u)\, f(v) \leqslant \frac{c^2}{\left(1 + \frac{r}{\sqrt{2}}\right)^k}. \tag{14}$$

in the domain of the second integral.

(v) Using the uniform controls on $|\partial_r L(f)|$ and $|\partial_r J(f)|$, prove that for all $(t,r) \in \mathbb{R}_+^2$:

$$\left|\Delta_t(r,r')\right| \leqslant e^{-\int_0^t L(f)(\tau,r)d\tau}\left|\Delta_0(r,r')\right| \tag{15}$$

$$+ \int_0^t e^{-\int_s^t L(f)(\tau,r)d\tau}\left|r'-r\right|(M_1 c + M_2)\, ds. \tag{16}$$

(vi) Taking for granted that there exists a constant $k > 0$ such that

$$L(f)(t,r) \geqslant k \tag{17}$$

for all $(t,r) \in \mathbb{R}_+^2$, deduce from (15) that for all $(t,r) \in \mathbb{R}_+^2$ we have:

$$\left|\Delta_t(r,r')\right| \leqslant \left|f_0(t,r) - f_0(t,r')\right| + \left|r'-r\right| \frac{M_1 c + M_2}{k}. \tag{18}$$

(vii) Use the previous point to prove the uniform continuity in r (uniformly in t) of the solutions of the Boltzmann equation.

Hint: Use the result of the first exercise.

Proof. (i) is just a rewriting of the terms involved. To prove (ii) we notice that the integrand in J and L are regular enough so that both $J(f)(t,\cdot)$ and $L(f)(t,\cdot)$ are C^1, so we can apply the Mean Value Theorem to get that for any t, r, r' there exists a value $\rho \in (r, r')$ such that

$$\begin{aligned}
&\partial_t \Delta_t(r,r') + L(f)(t,r)\,\Delta_t(r,r') = && (19)\\
&= J(f)(t,r) - J(f)(t,r') + f(t,r')\left(L(f)(t,r') - L(f)(t,r)\right) && (20)\\
&= (r'-r)\left(f(t,r')\,\partial_r L(f)(t,\rho) - \partial_r J(f)(t,\rho)\right). && (21)
\end{aligned}$$

Now to prove (iii) we get that using the fact that f is continuous and Leibniz rule the derivative of L is given by

$$\partial_r L(f)(t,r) = \partial_r\left[\int_0^r \left(2r + \frac{2u^2}{3r}\right) f(t,u)\,u^2 du + \int_r^{+\infty}\left(2u + \frac{2r^2}{3u}\right) f(t,u)\,u^2 du\right] = \tag{22}$$

$$\begin{aligned}
&= \left(2r + \frac{2r^2}{3r}\right) f(t,r)\,r^2 + \int_0^r \left(2 - \frac{2u^2}{3r^2}\right) f(t,u)\,u^2 du && (23)\\
&\quad - \left(2r + \frac{2r^2}{3r}\right) f(t,r)\,r^2 + \int_r^{+\infty}\left(2u + \frac{4r}{3u}\right) f(t,u)\,u^2 du && (24)\\
&= \int_0^r \left(2 - \frac{2u^2}{3r^2}\right) f(t,u)\,u^2 du + \int_r^{+\infty}\left(2u + \frac{4r}{3u}\right) f(t,u)\,u^2 du. && (25)
\end{aligned}$$

On the other hand we can write explicitly J as

$$\begin{aligned}
J(f)(t,r) = {}& 4\int_0^r \int_{\sqrt{r^2-v^2}}^r f(t,u)\,f(t,v)\,\frac{\sqrt{u^2+v^2-r^2}}{r} uv\,du\,dv && (26)\\
&+ 4\int_0^r \int_r^{+\infty} f(t,u)\,f(t,v)\,\frac{uv^2}{r} du\,dv && (27)\\
&+ 4\int_r^{+\infty} \int_0^r f(t,u)\,f(t,v)\,\frac{u^2 v}{r} du\,dv && (28)\\
&+ 4\int_r^{+\infty} \int_r^{+\infty} f(t,u)\,f(t,v)\,uv\,du\,dv. && (29)
\end{aligned}$$

Using again that f is continuous and Leibniz rule again we get

$$\partial_r J(f)(t,r) = 4\int_0^r f(t,u) f(t,r) u^2 du + 4\int_0^r f(t,r) f(t,v) v^2 dv \tag{30}$$

$$- 4\int_0^r \int_{\sqrt{r^2-v^2}}^r f(t,u) f(t,v) \frac{u^2+v^2}{r^2\sqrt{u^2+v^2-r^2}} uvdudv \tag{31}$$

$$+ 4\int_r^{+\infty} f(t,u) f(t,r) urdu - 4\int_0^r f(t,r) f(t,v) v^2 dv \tag{32}$$

$$- 4\int_0^r \int_r^{+\infty} f(t,u) f(t,v) \frac{uv^2}{r^2} dudv - 4\int_0^r f(t,u) f(t,r) u^2 du \tag{33}$$

$$+ 4\int_r^{+\infty} f(t,r) f(t,v) rvdv - 4\int_r^{+\infty}\int_0^r f(t,u) f(t,v) \frac{u^2 v}{r^2} dudv \tag{34}$$

$$- 4\int_r^{+\infty} f(t,u) f(t,r) urdu - 4\int_r^{+\infty} f(t,r) f(t,v) rvdv \tag{35}$$

$$= -4\int_0^r \int_{\sqrt{r^2-v^2}}^r f(t,u) f(t,v) \frac{u^2+v^2}{r^2\sqrt{u^2+v^2-r^2}} uvdudv \tag{36}$$

$$- 8\int_0^r \int_r^{+\infty} f(t,u) f(t,v) \frac{uv^2}{r^2} dudv \tag{37}$$

$$= -\frac{4}{r^2}\int_0^r \int_{\sqrt{r^2-v^2}}^r f(t,u) f(t,v) \frac{u^2+v^2}{r^2\sqrt{u^2+v^2-r^2}} uvdudv \tag{38}$$

$$- \frac{8}{r^2}\left(\int_0^r f(t,v) v^2 dv\right)\left(\int_r^{+\infty} f(t,u) udu\right). \tag{39}$$

To prove (iv) we first notice that for any $u \leqslant r$ we get

$$\left|\left(2 - \frac{2u^2}{3r^2}\right) u^2\right| \leqslant \left(2 + \frac{2}{3}\right) u^2 = \frac{8}{3}u^2, \tag{40}$$

and similarly for $u > r$ we get

$$\left|\left(2u + \frac{4r}{3u}\right) u^2\right| \leqslant \left(2u + \frac{4}{3}\right) u^2. \tag{41}$$

As a consequence we get that, using (14)

$$|\partial_r L(f)(t,r)| \leqslant \int_0^{+\infty}\left(\frac{8}{3} + 2u + \frac{4}{3}\right) f(t,u) u^2 du \leqslant C\int_0^{+\infty} \frac{u^2}{(1+u)^{k-1}} du \tag{42}$$

$$\leqslant M_1 < +\infty. \tag{43}$$

On the other hand, just notice that if $u^2+v^2 \geqslant r^2$, $u \leqslant r$ and $v \leqslant r$ we get

$$(1+u)(1+v) = 1 + u + v + uv \geqslant 1 + \max\{u,v\}. \tag{44}$$

Now, if $u \geqslant v$ we get $r^2 \leqslant u^2+v^2 \leqslant 2u^2$ and therefore $u \geqslant \frac{r}{\sqrt{2}}$; analogously, if $u < v$ and therefore $v \geqslant \frac{r}{\sqrt{2}}$ therefore we get that if $r^2 \leqslant u^2+v^2 \leqslant 2u^2$

$$(1+u)(1+v) \geqslant 1 + \frac{r}{\sqrt{2}}. \tag{45}$$

As a consequence using (14) we get

$$|\partial_r J(f)(t,r)| \leqslant 8\left(\int_0^r \frac{u^2}{r^2} f(t,u)\,du\right)\left(\int_r^{+\infty} v f(t,v)\,dv\right) \tag{46}$$

$$+4\int_0^r \int_{\sqrt{r^2-u^2}}^r \frac{uv\left(u^2+v^2\right)}{r^2\sqrt{u^2+v^2-r^2}} f(t,u)\, f(t,v)\,dvdu \tag{47}$$

$$\leqslant C\left(\int_0^{+\infty} \frac{1}{(1+u)^k}du\right)\left(\int_0^{+\infty} \frac{v}{(1+v)^k}dv\right) \tag{48}$$

$$+C\int_0^r \int_{\sqrt{r^2-u^2}}^r \frac{uv\left(u^2+v^2\right)}{r^2\sqrt{u^2+v^2-r^2}} \frac{1}{(1+u)^k(1+v)^k}dvdu \tag{49}$$

$$\leqslant C\left(\int_0^{+\infty} \frac{1}{(1+u)^k}du\right)\left(\int_0^{+\infty} \frac{v}{(1+v)^k}dv\right) \tag{50}$$

$$+C\frac{1}{\left(1+\frac{r}{\sqrt{2}}\right)^k}\int_0^r \int_{\sqrt{r^2-u^2}}^r \frac{uv\left(u^2+v^2\right)}{r^2\sqrt{u^2+v^2-r^2}}dvdu. \tag{51}$$

If we now change variables as $x=\frac{u^2}{r^2}$ and $y=\frac{v^2}{r^2}$ we get

$$\int_0^r \int_{\sqrt{r^2-u^2}}^r \frac{uv\left(u^2+v^2\right)}{r^2\sqrt{u^2+v^2-r^2}}dvdu = \int_0^1 \int_{\sqrt{1-(u')^2}}^1 \frac{2r^3}{\sqrt{(u')^2+(v')^2-1}}dv'du' = \frac{r^3}{8}. \tag{52}$$

As a consequence we get

$$|\partial_r J(f)(t,r)| \leqslant C\left(\int_0^{+\infty} \frac{1}{(1+u)^k}du\right)\left(\int_0^{+\infty} \frac{v}{(1+v)^k}dv\right) \tag{53}$$

$$+C'\frac{r^3}{(1+r)^k} \leqslant M_2 < +\infty. \tag{54}$$

From these estimates we deduce that

$$\partial_t \Delta_t\left(r,r'\right) + L(f)(t,r)\,\Delta_t\left(r,r'\right) \leqslant \left(r'-r\right)\left(f\left(t,r'\right)M_1 + M_2\right) \tag{55}$$

$$\leqslant \left(M_1 + cM_2\right)\left(r'-r\right). \tag{56}$$

Using Duhamel argument we get

$$e^{\int_0^t L(f)(\tau,r)d\tau}\left|\Delta_t\left(r,r'\right)\right| = \left|\Delta_0\left(r,r'\right) + \int_0^t e^{\int_0^s L(f)(\tau,r)d\tau}\left(\partial_t \Delta_t\left(r,r'\right)\right.\right. \tag{57}$$

$$\left.\left. + L(f)\left(r,r'\right)\Delta_t\left(r,r'\right)\right)ds\right| \tag{58}$$

$$\leqslant \left|\Delta_0\left(r,r'\right)\right| + \int_0^t e^{\int_0^s L(f)(\tau,r)d\tau}\left(M_1+cM_2\right)\left|r'-r\right|ds, \tag{59}$$

which implies (15).

Finally, to conclude that f is absolutely continuous, we use the fact that f is continuous and decays at infinity to apply the previous exercise and conclude.

□